Gaussian Approximation to Ising (φ^4) field Theory

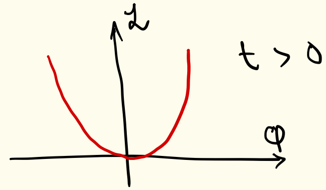
As just discussed, the Ising field theory is

$$Z = \int D\varphi \; e^{-\int d^d r \left[\frac{(\nabla\varphi)^2}{2} + t\varphi^2 + u\varphi^4 + h\varphi \right]}$$

There is no known way to evaluate this path-integral exactly. The best way to make progress is to perform an RG calculation by introducing a small parameter, which we will soon do.

But before that, let's consider an approximate way to proceed, which we will call 'Gaussian approximation' to Z. The basic idea is to first drop the $(\nabla\varphi)^2$ term so that one recovers Landau theory. Within Landau theory, for $t > 0$, the potential at $h = 0$

$|t|\varphi^2 + u\varphi^4$:

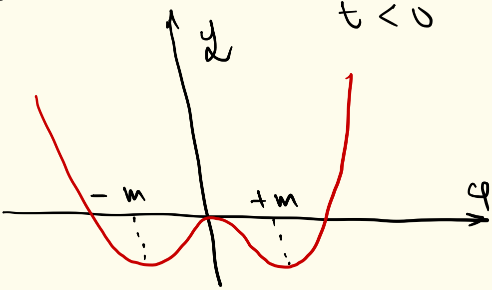


the u term doesn't play an essential role for $t > 0$ since $\varphi^2 \gg \varphi^4$ when φ is small. Thus, it's a reasonable approximation away from criticality to write $\mathcal{L} \approx \frac{(\nabla\varphi)^2}{2} + t\varphi^2 + h\varphi$.

For $t < 0$ however, one can't drop the 'u' term since the potential is otherwise unbounded. Indeed the magnetization within the Landau theory is given by

$$m = \varphi^* \sim \sqrt{-\frac{t}{u}}$$ which is ill-defined at $u = 0$.

The potential looks like:



One can still make a gaussian approximation for $t<0$ by expanding $\mathcal{L}(\varphi)$ around one of the minimas eg. $\varphi = m + \delta\varphi$ and keeping terms to only quadratic order in $\delta\varphi$.

Lets study the approach to criticality and associated critical exponents within the Gaussian approximation. We will focus on $t > 0$ and leave the case $t < 0$ for homework.

The simplest way to evaluate Z_{gauss} = the partion f^n within Gaussian approximation is to perform a Fourier transform on φ and work in momentum space.

$$\varphi(\vec{r}) = \frac{1}{V}\sum_k \varphi(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

$$\left(\text{inverse}: \quad \varphi(\vec{k}) = \int_{\vec{r}} \varphi(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\right)$$

The 'action' $S_{gauss} = \int d^d r\, \mathcal{L}_{gauss}$

$$= \int d^d r \left[\frac{(\nabla\varphi)^2}{2} + t\varphi^2(r) + h\varphi(r)\right]$$

$$= \frac{1}{V}\sum_k \left[\frac{k^2}{2}\varphi_k\varphi_{-k} + t\,\varphi_k\varphi_{-k}\right] + h\,\varphi(k=0)$$

$$= \sum_k \frac{|\varphi_k|^2}{2V}(k^2+2t) + h\,\varphi(k=0).$$

Thus, the system resembles decoupled harmonic oscillators. From equipartition theorem (or, a direct Gaussian integral evaluation),

$$\frac{\langle|\varphi_k|^2\rangle(k^2+2t)}{2V} = 2\times\frac{1}{2} = 1$$

φ_k is complex $\equiv$ two real modes.

$$\Rightarrow \quad \langle |\varphi_k|^2 \rangle = \frac{V}{2t + k^2} \qquad (k \neq 0).$$

The $\varphi_{k=0}$ mode can be dealt with separately.

$$\langle \varphi_{k=0} \rangle = \frac{hV}{2t}$$

Lets calculate real-space correlation f^n $\langle \varphi(\vec{r})\, \varphi(\vec{r}') \rangle$ to obtain exponents η and ν:

$$\langle \varphi(r)\, \varphi(r') \rangle = \frac{1}{V^2} \sum_{kk'} \langle \varphi_k e^{ik.r} \varphi_{k'} e^{ik'.r'} \rangle$$

$$= \frac{1}{V^2} \sum_{kk'} \langle |\varphi_k|^2 \rangle \, \delta_{k+k'} \, e^{ik.r + ik'.r'}$$

$$= \frac{1}{V^2} \sum_{k} \frac{V}{2t+k^2} \, e^{ik.(r-r')}$$

$$= \frac{1}{V} \frac{V}{(2\pi)^d} \int \frac{d^dk}{2t+k^2} \, e^{ik.(r-r')}$$

$$= \frac{1}{(2\pi)^d} \int \frac{d^dk \; e^{ik.(r-r')}}{2t+k^2}$$

By dimension analysis,

$$\langle \varphi(r)\, \varphi(r') \rangle \sim \frac{e^{-|r-r'|\sqrt{2t}}}{r^{d-2}}$$

Therefore, right at the critical point,

$$\langle \varphi(r)\, \varphi(r') \rangle \sim \frac{1}{r^{d-2}}$$

Comparing it with the definition of critical exponent η : $\langle \varphi(r)\, \varphi(r') \rangle = \frac{1}{r^{d-2+\eta}}$

$\Rightarrow$ $\eta = 0$ within the Gaussian approximation.

Sidenote :

It turns out that η is a good measure of non-Gaussianity of a critical point, e.g. for a quantum theory, $\langle \varphi(k,\omega)\, \varphi(-k,-\omega) \rangle \sim \frac{1}{(\omega^2 - k^2)^{1-\frac{\eta}{2}}}$

$\eta \neq 0$ implies that the two-point f^n doesn't have a pole $\Rightarrow$ doesn't have well defined particles.

Slightly away from the critical point,

$$\langle \varphi(r)\, \varphi(r') \rangle \sim \frac{e^{-|r-r'|\sqrt{2t}}}{r^{d-2}}$$

$$\sim \frac{e^{-|r-r'|/\xi}}{r^{d-2}}$$

where $\xi \sim \dfrac{1}{\sqrt{2t}}$ defines a divergent correlation length as $t \to 0$. Recall the definition of critical exponent ν:

$$\xi \sim |t|^{-\nu}.$$

Therefore, within the Gaussian approximation, exponent $\nu = \frac{1}{2}$.

Next, let's calculate the free energy itself so that we may obtain thermodynamic quantities such as specific heat etc.

Free energy within Gaussian approximation (at $h=0$)

$$e^{-\beta F} = Z_{Gauss} = \int D\varphi_k \; e^{-\sum_k \frac{|\varphi_k|^2}{2V}(k^2+2t)}$$

where $D\varphi_k \equiv D\,\mathrm{Re}(\varphi_k)\; D(\mathrm{Im}\,\varphi_k)$.

However, there is a subtlety here: $\varphi(r)$ is real $\Rightarrow$ $\varphi(k) = \varphi^*(-k)$. Therefore, the set $\{\mathrm{Re}(\varphi_k)\}$, and $\{\mathrm{Im}(\varphi_k)\}$ are not independent. To correctly count the degrees of freedom, we should restrict to half the phase-space, say, $k_x > 0$. Alternatively, we may not put any such constraint on the integral, but simply divide the final answer for the free energy by $\frac{1}{2}$.

Following this latter approach,

$$-\beta F = \frac{1}{2}\sum_k \log\left(\frac{2\pi V}{k^2+2t}\right)$$

$$\Rightarrow F = -\frac{T}{2}\sum_k \log\left(\frac{2\pi V}{k^2+2t}\right)$$

Lets calculate the specific heat using this form.

$$C = -T \frac{\partial^2 F}{\partial T^2}$$

As one may expect in the expression for F above, the T dependence in $\sum_k \log\left(\frac{2\pi V}{k^2+2t}\right)$ is the leading source of any potential singularity and the prefactor $-\frac{T}{2}$ can be set to $-\frac{T_c}{2}$ for our purposes (recall $t = \frac{T-T_c}{T_c}$).

Thus, C close to T_c is

$$C \simeq -T_c \frac{\partial^2 F}{\partial T^2} = -\frac{1}{T_c} \frac{\partial^2 F}{\partial t^2}$$

$$= -\frac{1}{T_c} \times -\frac{T_c}{2} \sum_k \frac{4}{(k^2+2t)^2}$$

$$= 2 \sum_k \frac{1}{(k^2+2t)^2} = \frac{2V}{(2\pi)^d} \int_0^\Lambda \frac{d^d k}{(k^2+2t)^2}$$

where we have set the upper limit of integration to $\Lambda \sim (\text{lattice spacing})^{-1}$, and the lower limit to 0 (if we put the system in a box of size L^d, the lower limit would be $1/L$ instead).

We are primarily interested in the divergence due to the lower limit (i.e. $k \to 0$) as that reflects the long-distance critical fluctuations.

For $d < 4$, this integral diverges as

$C \sim t^{(d-4)/2}$ while for $d > 4$, it converges. Therefore, we conclude that

$$C \sim t^{-(2-d/2)} \quad , \quad d < 4$$

$$\sim t^{0} \quad , \quad d > 4$$

Recalling the critical exponent α as $C \sim t^{-\alpha}$

$\Rightarrow \quad \alpha = 2 - \dfrac{d}{2}$ for $d < 4$ and $\alpha = 0$ for $d > 4$ within the Gaussian approximation.

Contrast this with the result $\alpha = 0$ obtained within the Landau theory / mean-field theory without Gaussian fluctuations.